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$\mathcal{F}$-mixing and weak disjointness ☆

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Abstract

A topological system (X, f) is $\mathcal{F}$-transitive if for each pair of opene subsets U and V of X, $N_f(U, V) = \{n \in \mathbb{Z}_+\colon f^n U \cap V \neq \emptyset\} \in \mathcal{F}$, where $\mathcal{F}$ is a collection of subsets of $\mathbb{Z}_+$ which is hereditary upward. (X, f) is $\mathcal{F}$-mixing if $(X \times X, f \times f)$ is $\mathcal{F}$-transitive. In this paper $\mathcal{F}$-mixing systems are characterized in terms of the chaoticity of the systems. Moreover, weak disjointness is studied via family. We will give conditions such that a dual theorem of the Weiss–Akin–Glasner theorem holds. Examples with this dual theorem fails for some "good" families are obtained.

MSC: 37

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Introduction

By a *topological dynamical system* (TDS, for short) we mean a pair (X, f), where X is a compact metric space and f a surjective continuous map from X to itself. The notion of disjointness of two TDS was introduced by Furstenberg [5], and a weaker notion, namely weak disjointness appeared later in [11]. The two notions are very much related at least for minimal systems. Let (X, f) be a TDS. (X, f) is (topologically) *transitive* if for each pair of opene (i.e., open and nonempty) subsets U, V there is some $n \geqslant 1$ such that $f^n(U) \cap V \neq \emptyset$. Set $N_f(U, V) = \{n \in \mathbb{Z}_+\colon f^n U \cap V \neq \emptyset\}$. Two TDS are said to be *weakly disjoint* if their product is transitive. Recently the authors in [4] gave a notion known as scattering. Though this notion was introduced using the complexity of open covers, it is

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equivalent to the statement that it is weakly disjoint from all minimal systems. This adds the evidence to the fact that weak disjointness appears naturally in the study of dynamical systems.

The duality question is a natural question related to weak disjointness. Let P be a dynamical property and let $P^{\curlywedge}$ be the dynamical property such that a system has $P^{\curlywedge}$ if and only if it is weakly disjoint from any system having P. It is known $P^{\curlywedge\curlywedge\curlywedge} = P^{\curlywedge}$ [8]. Thus it is interesting to know the dynamical properties P_1 and P_2 for which $P_1^{\curlywedge} = P_2$ and $P_2^{\curlywedge} = P_1$. To do this we need to know a way to describe $P^{\curlywedge}$ when P is known. It turns out that the *family* notion (see Section 1) is a useful tool.

A family is a collection of subsets of $\mathbb{Z}_+$ which is hereditary upward. The idea of family can be traced back to Gottschalk and Hedlund [7]. It was exploited by Furstenberg [6] and was systematically treated by Akin [3]. Let us see how it can be used to describe some known dynamical properties. Let (X, f) be a TDS. (X, f) is (topologically) *weakly mixing* if $(X \times X, f \times f)$ is transitive, and it is (topologically) *strongly mixing* if for each pair of opene subsets U, V there is $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for each $n \geqslant N$. Thus we see that (X, f) is transitive iff $N_f(U, V)$ is infinite, it is weakly mixing iff $N_f(U, V)$ is thick (i.e., containing arbitrarily long intervals of integers) [5] and it is strong mixing iff $N_f(U, V)$ is cofinite for each pair of opene subsets U, V of X.

The main goal of the paper is to study duality question using family notion. Generally $(P^{\curlywedge})^{\curlywedge} = P$ does not hold. For examples, the systems which are weakly disjoint from all strongly mixing systems are the transitives, but a system which is weakly disjoint from all transitive systems needs not to be strongly mixing. In the case when P is $\mathcal{F}$-transitivity we will give some conditions such that $(P^{\curlywedge})^{\curlywedge} = P$ holds. Using the notion of $\mathcal{F}$-mixing, we will give a characterization of such a property using the chaoticity of the system.

We organize the paper as follows. In Section 1 we introduce the notations related to family. In Section 2 we discuss $\mathcal{F}$-mixing and give a characterization of the property. We study weak disjointness and the duality questions related to it in Section 3, and give conditions such that a dual theorem of the Weiss–Akin–Glasner theorem holds. At the last section we analyze examples with this dual theorem fails for some "good" families.

1. Preliminary

Firstly we recall some notations related to a family (for details see [6,3,2]). For a nonempty set A, denote by $\mathcal{P}(A)$ the collection of all subsets of A. For simplicity let $\mathcal{P} = \mathcal{P}(\mathbb{Z}_+)$, where $\mathbb{Z}_+$ is the set of non-negative integers. A subset $\mathcal{F}$ of $\mathcal{P}$ is a *family*, if it is hereditary upwards. That is, $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. A family $\mathcal{F}$ is *proper* if it is a proper subset of $\mathcal{P}$, i.e., neither empty nor all of $\mathcal{P}$. It is easy to see that $\mathcal{F}$ is proper if and only if $\mathbb{Z}_+ \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$. Any subset $\mathcal{A}$ of $\mathcal{P}$ can generate a family $[\mathcal{A}] = \{F \in \mathcal{P}\colon F \supset A \text{ for some } A \in \mathcal{A}\}$. If a proper family $\mathcal{F}$ is closed under intersection, then $\mathcal{F}$ is called a *filter*. For a family $\mathcal{F}$, the *dual family* is

$$k\mathcal{F} = \{F \in \mathcal{P} \mid \mathbb{Z}_+ \setminus F \notin \mathcal{F}\} = \{F \in \mathcal{P} \mid F \cap F' \neq \emptyset \text{ for all } F' \in \mathcal{F}\}.$$

$k\mathcal{F}$ is a family, proper if $\mathcal{F}$ is. Clearly,

$$k(k\mathcal{F}) = \mathcal{F} \quad \text{and} \quad \mathcal{F}_1 \subset \mathcal{F}_2 \implies k\mathcal{F}_2 \subset k\mathcal{F}_1. \tag{1.1}$$

For families $\mathcal{F}_1$ and $\mathcal{F}_2$, let $\mathcal{F}_1 \cdot \mathcal{F}_2 = \{F_1 \cap F_2 \mid F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$. Thus we have $\mathcal{F}_1 \cup \mathcal{F}_2 \subset \mathcal{F}_1 \cdot \mathcal{F}_2$. It is easy to check that $\mathcal{F}$ is a filter iff $\mathcal{F} = \mathcal{F} \cdot \mathcal{F}$. Also, $\mathcal{F}_1 \cdot \mathcal{F}_2$ is proper iff $\mathcal{F}_2 \subset k\mathcal{F}_1$.

For $i \in \mathbb{Z}_+$ let $g^i : \mathbb{Z}_+ \to \mathbb{Z}_+$ be defined by $g^i(j) = i + j$. A family $\mathcal{F}$ is called *translation invariant* if for every $i \in \mathbb{Z}_+$, $F \in \mathcal{F} \Longleftrightarrow g^{-i}(F) \in \mathcal{F}$. For a family $\mathcal{F}$ let

$$\tau\mathcal{F} = \left\{ F \in \mathcal{P} \mid \bigcap_{j=1}^{n} g^{-i_j}(F) \in \mathcal{F} \quad \text{for } n \in \mathbb{N} \text{ and each } \{i_1, i_2, \ldots, i_n\} \subset \mathbb{Z}_+ \right\}. \tag{1.2}$$

$\mathcal{F}$ is a *thick family* if and only if $\tau\mathcal{F} = \mathcal{F}$, and it is easy to see $\tau\mathcal{F}$ is the largest thick family contained in $\mathcal{F}$. Let $\mathcal{B}$ the family of all infinite subsets of $\mathbb{Z}_+$. It is easy to see that $\mathcal{B}$ is the largest proper translation invariant family and its dual $k\mathcal{B}$, the family of cofinite subset, is the smallest one.

A subset F of $\mathbb{Z}_+$ is *thick* if $F \in \tau\mathcal{B}$, equivalently, F is thick if and only if it contains arbitrarily long runs of positive integers. Each element of $k\tau\mathcal{B}$ is said to be *syndetic* or *relatively dense*. F is syndetic if and only if there is N such that $\{i, i+1, \ldots, i+N\} \cap F \neq \emptyset$ for every $i \in \mathbb{Z}_+$. The set in $\tau k\tau\mathcal{B}$ is called *replete* or *thickly syndetic*. $F \in \tau k\tau\mathcal{B}$ if and only if for every N the positions where length N runs begin form a syndetic set. The set in $k\tau k\tau\mathcal{B}$ is called *big* or *piecewise syndetic*. $F \in k\tau k\tau\mathcal{B}$ if and only if it is the intersection of a thick set and a syndetic set. All of these families are translation invariant, and $\tau k\tau\mathcal{B}$ is a filter.

Let A be a subset of either $\mathbb{Z}_+$ or $\mathbb{Z}$. The *upper Banach density of A* is

$$d^*(A) = \limsup_{|I| \to \infty} \frac{|A \cap I|}{|I|}, \tag{1.3}$$

where I ranges over intervals of $\mathbb{Z}_+$ or $\mathbb{Z}$ and $|\cdot|$ denote the cardinality of the set. The *upper density* of a subset A of $\mathbb{Z}_+$ is

$$\bar{d}(A) = \limsup_{N \to \infty} \frac{|A \cap \{0, 1, \ldots, N-1\}|}{N-1}. \tag{1.4}$$

(If A is subset of $\mathbb{Z}$, then $\bar{d}(A) = \limsup_{N \to \infty} \frac{|A \cap \{-N, -N+1, \ldots, N\}|}{2N+1}$.) The *lower Banach density* $d_*(A)$ and the *lower density* $\underline{d}(A)$ are similarly defined. If $\bar{d}(A) = \underline{d}(A)$, then we say A has density $d(A)$. Using density we can define lots of interesting families which we will introduce in the sequel.

Let (X, f) be a dynamical system and $A, B \subset X$. We define the *hitting time set*

$$N_f(A, B) = \left\{ n \in \mathbb{Z}_+ \mid f^n(A) \cap B \neq \emptyset \right\}. \tag{1.5}$$

A topological system (X, f) is $\mathcal{F}$*-transitive* if for each pair of opene subsets U and V of X $N_f(U, V) \in \mathcal{F}$. (X, f) is $\mathcal{F}$*-mixing* if $(X \times X, f \times f)$ is $\mathcal{F}$-transitive.

We say a family $\mathcal{F}$ can be *realized by TDS* if for every element A in $\mathcal{F}$ there is some $\mathcal{F}$-transitive system (X, f) and some opene subsets U, V of X such that $N_f(U, V) \subset A$. Set

$$r\mathcal{F} = \bigcup_{\substack{(X,f) \text{ is} \\ \mathcal{F}\text{-transitive}}} \mathcal{T}_{(X,f)}, \tag{1.6}$$

where $\mathcal{T}_{(X,f)} = [\{N_f(U,V)\colon\ U,V \text{ are opene subsets of } X\}]$. For any family $\mathcal{F}$, by the definition we have $\mathcal{F}$,

- $r\mathcal{F}$ can be realized by TDS, and
- a system is $\mathcal{F}$-transitive iff it is $r\mathcal{F}$-transitive, and
- $\mathcal{F}_1$-transitivity equals to $\mathcal{F}_2$-transitivity iff $r\mathcal{F}_1 = r\mathcal{F}_2$.

If a system (X, f) is $\mathcal{F}$-transitive we will write $(X, f) \in r\mathcal{F}$.

Proposition 1.1. *Let $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ be proper families. Then*

(1) *$r\mathcal{F}$ is a translation invariant family.*
(2) $\mathcal{F}_1 \subset \mathcal{F}_2 \Longrightarrow r\mathcal{F}_1 \subset r\mathcal{F}_2$.
(3) $rr\mathcal{F} = r\mathcal{F}$.
(4) *For any family $\mathcal{G}$ with $r\mathcal{F} \subset \mathcal{G} \subset \mathcal{F}$, we have $r\mathcal{G} = r\mathcal{F}$.*
(5) *If $\mathcal{F}_1$ and $\mathcal{F}_2$ can be realized by TDS, so is $\mathcal{F}_1 \cdot \mathcal{F}_2$. In general for any $\mathcal{F}_1, \mathcal{F}_2$*

$$r(\mathcal{F}_1 \cdot \mathcal{F}_2) \supset r(r\mathcal{F}_1 \cdot r\mathcal{F}_2) = r\mathcal{F}_1 \cdot r\mathcal{F}_2.$$

Proof. (2), (3) and (4) are obvious. First we show (1), i.e., for every $n \in \mathbb{Z}_+$, $F \in r\mathcal{F} \Longleftrightarrow g^{-n}(F) \in r\mathcal{F}$.

If $F \in r\mathcal{F}$, then there is some $\mathcal{F}$-transitive system (X, f) and some opene subsets U, V of X such that $N_f(U,V) \subset F$. Hence $N_f(U, f^{-n}(V)) = g^{-n}(N_f(U,V)) \subset g^{-n}(F)$. Since $N_f(U, f^{-n}(V)) \in r\mathcal{F}$ and $r\mathcal{F}$ is hereditary upwards, $g^{-n}(F) \in r\mathcal{F}$.

Now assume $g^{-n}(F) \in r\mathcal{F}$. Then there is some $\mathcal{F}$-transitive system (X, f) and some opene subsets U, V of X such that $N_f(U,V) \subset g^{-n}(F)$. Let $h:(\widetilde{X}, \tilde{f}) \to (X, f)$ be the natural extension of (X, f), i.e., $\widetilde{X} = \{(x_1, x_2, \ldots)\colon\ f(x_{i+1}) = x_i,\ x_i \in X,\ i \in \mathbb{N}\}$ which is a subspace of the product space $\prod_{i=1}^{\infty} X$ with the compatible metric d_T defined by $d_T((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^i}$, $\tilde{f}: \widetilde{X} \to \widetilde{X}$ is the shift homeomorphism, i.e., $\tilde{f}(x_1, x_2, \ldots) = (f(x_1), x_1, x_2, \ldots)$ and $h(x_1, x_2, \ldots) = x_1$.

Then $(\widetilde{X}, \tilde{f})$ is also $\mathcal{F}$-transitive. Moreover,

$$N_{\tilde{f}}\big(h^{-1}(U), h^{-1}(V)\big) = N_f(U,V) \subset g^{-n}(F).$$

Since $\tilde{f}$ is a homeomorphism,

$$N_{\tilde{f}}\big(\tilde{f}^{-n}\big(h^{-1}(U)\big), h^{-1}(V)\big) = g^{-(-n)}\big(N_f(U,V)\big) \subset F.$$

As $(\widetilde{X}, \tilde{f})$ is $\mathcal{F}$-transitive, $F \in r\mathcal{F}$ by the definition.

Now we show (5). For this purpose let $F \in r\mathcal{F}_1 \cdot r\mathcal{F}_2$. Then $F = F_1 \cap F_2$ with $F_i \in r\mathcal{F}_i$, $i = 1, 2$. Hence there are TDS (X_i, f_i) which is $\mathcal{F}_i$-transitive and opene sets U_i, V_i of X_i such that $N_{f_i}(U_i, V_i) \subset F_i$, $i = 1, 2$.

As $r\mathcal{F}_1 \cdot r\mathcal{F}_2$ is a family, it is easy to see that $f_1 \times f_2$ is $r\mathcal{F}_1 \cdot r\mathcal{F}_2$-transitive. Moreover,

$$N_{f_1 \times f_2}(U_1 \times U_2, V_1 \times V_2) \subset F_1 \cap F_2 = F.$$

Thus $r(r\mathcal{F}_1 \cdot r\mathcal{F}_2) = r\mathcal{F}_1 \cdot r\mathcal{F}_2$. This implies $r(\mathcal{F}_1 \cdot \mathcal{F}_2) \supset r(r\mathcal{F}_1 \cdot r\mathcal{F}_2) = r\mathcal{F}_1 \cdot r\mathcal{F}_2$. $\square$

Generally speaking, $\mathcal{F} \neq r\mathcal{F}$, i.e., for a family $\mathcal{F}$ not every element of it can be realized by a TDS. But there do exist cases when $\mathcal{F} = r\mathcal{F}$. The following W-AG Lemma illustrates this situation:

Weiss–Akin–Glasner Lemma [2]. *Let $\mathcal{F}$ be a proper, translation invariant, thick family and let $A \in \mathcal{F}$. Then there exists a system (X, f) which is $\mathcal{F}$-transitive and there is an opene set U in X such that $N_f(U, U) = A \cup \{0\}$.*

Hence every proper translation invariant thick family $\mathcal{F}$ can be realized by TDS, i.e., $r\mathcal{F} = \mathcal{F}$. And in this case if a system (X, f) is $\mathcal{F}$-transitive we also denote it by $(X, f) \in \mathcal{F}$.

In the rest of the section we generalize the definition of $\mathcal{F}$-mixing. We say (X, f) has *double property* (DP for short), if $(X \times X, f \times f)$ has property P, where P is a dynamical property. If property P_1 is stronger than P_2, then we denote it by $P_1 \geqslant P_2$. If a system (X, f) has property P, denote it by $(X, f) \in P$.

Proposition 1.2. *Let P, P_1 and P_2 be properties which is inherited by factors. Then*

(1) $DP \geqslant P$;
(2) *If $P_1 \geqslant P_2$, then $DP_1 \geqslant DP_2$;*
(3) *If $D(DP) = DP$, then for every property P' with $P \leqslant P' \leqslant DP$ we have $DP' = DP$.*

Proof. (1) Let $\pi : (X \times X, f \times f) \to (X, f)$ and $(X \times X, f \times f)$ have property P. Since P is inherited by factors, (X, f) has property P.

(2) Obvious.

(3) On one hand, $P \leqslant P'$ implies $DP' \geqslant DP$. On the other hand, $P' \leqslant DP$ implies $DP = DDP \geqslant DP'$. So we have $DP = DP'$. $\square$

Now we discuss some examples:

Example 1.3. Let P and P' be dynamical properties.

(1) If P is $\mathcal{F}$-transitivity, then DP stands for $\mathcal{F}$-mixing.
(2) If $P = (P')^{\curlywedge}$, then (X, f) has DP iff $(X \times X, f \times f)$ is weakly disjoint from any system having P'.
(3) If P is minimality, then (X, f) has DP iff (X, f) is trivial.
(4) If P is semi-simplity, i.e., $(X, f) \in P$ iff every point in X is minimal, then $(X, f) \in DP$ iff (X, f) is distal.
(5) A system (X, f) is called n-rigid if the product system of n copies of (X, f) is pointwise recurrent [1]. If P is 1-rigidity, then $DP = 2$-rigidity and generally $D^n P = D(D^{n-1} P) = 2^n$-rigidity. We believe this offers an example of a property P satisfying $P \lneqq DP \lneqq DDP \lneqq D^3 P \lneqq \cdots$, though it is an open question.

In Section 2 we will discuss DP in the case $P = \mathcal{F}$-transitivity.

2. $\mathcal{F}$-mixing

In this section we study $\mathcal{F}$-mixing. As $\mathcal{F}$-transitivity is a residual property, so is $\mathcal{F}$-mixing, i.e., it is inherited by factors, almost one-to-one lifts and surjective inverse limits (see [2, Theorem 1.3 and Theorem 2.3]). First we recall some classical results on weak mixing [3,5,10].

Theorem 2.1. *Let (X, f) be a dynamical system. Then the following statements are equivalent*:

(1) *(X, f) is weakly mixing, i.e., $(X \times X, f \times f)$ is transitive.*
(2) *$N_f(U, U) \cap N_f(U, V) \in \mathcal{B}$, for all opene sets U, V in X.*
(3) *For all opene sets U_1, U_2, V_1, V_2 in X, there exist opene sets U, V in X such that $N_f(U, V) \subset N_f(U_1, V_1) \cap N_f(U_2, V_2)$.*
(4) *$\{N_f(U, V) \mid U, V$ are opene sets in $X\}$ generates a filter.*
(5) *(X, f) is $\tau\mathcal{B}$-transitive.*

A family $\mathcal{F}$ is *full* if $\mathcal{F} \cdot k\mathcal{F} \subset \mathcal{B}$. If $\mathcal{F}$ is full then $k\mathcal{B} \subset \mathcal{F} \subset \mathcal{B}$. If $\mathcal{F}$ is a filter, then $k\mathcal{B} \subset \mathcal{F}$ implies $\mathcal{F}$ is full (see [3]).

Theorem 2.2 [3]. *Let (X, f) be a dynamical system and $\mathcal{F}$ be a full family. Then the following statements are equivalent*:

(1) *(X, f) is $\mathcal{F}$-mixing.*
(2) *(X, f) is $\tau\mathcal{F}$-transitive.*
(3) *(X, f) is $\mathcal{F}$-transitive and weakly mixing.*
(4) *There exists a translation invariant filter $\mathcal{F}' \subset \mathcal{F}$ such that (X, f) is $\mathcal{F}'$-transitive.*

An immediate consequence is

Proposition 2.3. *Let (X, f) be a dynamical system and $\mathcal{F}$ be a full family. If (X, f) is $\mathcal{F}$-mixing, then for every $n \in \mathbb{N}$, $(X^{(n)}, f^{(n)})$ is $\mathcal{F}$-transitive, where $X^{(n)} = X \times \cdots \times X$ (n times) and $f^{(n)} = f \times \cdots \times f$ (n times).*

Proof. For every opene sets $U_1, U_2, \ldots, U_n, V_1, V_2, \ldots, V_n$ of X, we have $N(U_1 \times U_2 \times \cdots \times U_n, V_1 \times V_2 \times \cdots \times V_n) = \bigcap_{i=1}^n N_f(U_i, V_i) \in \mathcal{F}$ by Theorem 2.2(4). $\square$

When P is $\mathcal{F}$-trans we may say something concerning DP.

Theorem 2.4. *Let $\mathcal{F}$ be a full family.*

(1) *If P is $\mathcal{F}$-transitivity, then $D^2P = DP = \mathcal{F}$-mixing.*
(2) *Let P be a property with $\mathcal{F}$-trans $\leqslant P \leqslant \mathcal{F}$-mixing. Then $DP = \tau\mathcal{F}$-trans $= \mathcal{F}$-mixing.*

Proof. (1) As it is obvious that $D^2P \geqslant DP$, it remains to show $DP \geqslant D^2P$. Assume that (X, f) has DP, i.e., $\mathcal{F}$-mixing. By Proposition 2.3 we know $(X^{(4)}, f^{(4)})$ is $\mathcal{F}$-trans. So by definition $(X^{(2)}, f^{(2)})$ is $\mathcal{F}$-mixing, i.e., DP. That is (X, f) has $D(DP) = D^2P$. Hence $DP \geqslant D^2P$.

(2) Apply Proposition 1.2(3). □

By W-AG Lemma we can distinguish two different $\mathcal{F}$-mixing systems:

Proposition 2.5. *If $\mathcal{F}_1 \subsetneq \mathcal{F}_2$ are two proper, translation invariant, thick families, then there exists a system which is $\mathcal{F}_2$-transitive* (*at the same time $\mathcal{F}_2$-mixing*) *but not $\mathcal{F}_1$-transitive* (*$\mathcal{F}_1$-mixing*).

Proof. Let $A \in \mathcal{F}_2 \setminus \mathcal{F}_1$. By W-AG Lemma there exists a system (X, f) which is $\mathcal{F}_2$-transitive and there is an opene set U in X such that $N_f(U,U) = A \cup \{0\} \in \mathcal{F}_2$. As $N_f(U,U) \notin \mathcal{F}_1$, (X, f) is not $\mathcal{F}_1$-transitive. □

In [13] the authors obtained some equivalence conditions for weak mixing and strong mixing systems from viewpoint of chaoticity. Now we use Xiong–Yang's lemma to get a similar equivalent condition for $\mathcal{F}$-mixing systems.

Definition 2.6. Suppose $f : X \to X$ is a continuous map and $F \subset \mathbb{Z}_+$. A subset C of X is a *chaotic set with respect to F* if for any subset A of C and for any continuous map $g : A \to X$ there is a subsequence $\{q_i\} \subset F$ such that $\lim_{i\to\infty} f^{q_i}(x) = g(x)$ for every $x \in A$, i.e., $\{f^n|_A\colon n \in F\}$ is pointwise dense in $C(A, X)$.

Lemma 2.7 (Xiong–Yang). *Let $f : X \to X$ be a continuous map, where X is a separable locally compact metric space containing at least two points, and let $F \subset \mathbb{N}$. If for any opene sets $A_1, A_2, \dots, A_m$ and $B_1, B_2, \dots, B_n$ and for any $N > 0$ there is $p \in F \cap (N, +\infty)$ such that $f^p(A_i) \cap B_j \neq \emptyset$ for any $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ (i.e., $(\bigcap_{i,j} N_f(A_i, B_j)) \cap F \cap (N, +\infty) \neq \emptyset$), then there is a c-dense F_σ-subset C of X which is chaotic with respect to F.*

Theorem 2.8. *Let (X, f) be a dynamical system, where X is a separable locally compact metric space containing at least two points and $\mathcal{F}$ be a full family. Then (X, f) is $\mathcal{F}$-mixing if and only if for any $F \in k\mathcal{F}$ there is c-dense F_σ-subset C of X which is chaotic with respect to F.*

Proof. If (X, f) is $\mathcal{F}$-mixing, then by Theorem 2.1, 2.2 and Lemma 2.7 we get the conclusion. Now we assume for any $F \in k\mathcal{F}$, there is C satisfying the conditions mentioned in the theorem. For any opene subsets U, V of $X \times X$ we choose $(x_1, x_2) \in U \cap (C \times C)$ and $(y_1, y_2) \in V$. By the definition of C, there is a subsequence $\{q_i\} \subset F$ such that $\lim_{i\to\infty} f^{q_i}(x_1) = y_1$ and $\lim_{i\to\infty} f^{q_i}(x_2) = y_2$. Hence $N_{f\times f}(U, V) \cap F \neq \emptyset$. This implies that $N_{f\times f}(U, V) \in kk\mathcal{F} = \mathcal{F}$. So (X, f) is $\mathcal{F}$-mixing. □

We say that a dynamical system (X, f) is *spatiotemporally chaotic* or *ST chaotic* for short if it is transitive and for any $x \in X$ and every neighborhood U of x there is $y \in U$ such that (x, y) is Li–Yorke pair, i.e.,

$$\liminf_{n\to+\infty} d\bigl(f^n(x), f^n(y)\bigr) = 0, \qquad \limsup_{n\to+\infty} d\bigl(f^n(x), f^n(y)\bigr) > 0.$$

The following theorem is proved in [9], here we give another proof using Theorem 2.8.

Theorem 2.9. *Any strongly mixing system (X, f) is ST chaotic.*

Proof. For any $x \in X$ we show there is a dense set C such that for any $y \in C$, (x, y) is a Li–Yorke pair. Let $f^{n_i}(x) \to x_1$, where $n_i \to \infty$. Choose $x_2 \in X$ with $x_2 \neq x_1$ and $d(x_1, x_2) > \delta$, where $\delta < \operatorname{diam}(X)/2$. By Theorem 2.8 there is a c-dense F_σ-subset C of X which is chaotic with respect to $\{n_i\}$. Let $g_1, g_2 \in C(C, X)$ such $g_1 \equiv x_1$ and $g_2 \equiv x_2$. Then there are $\{n'_i\}, \{n''_i\} \subset \{n_i\}$ such that for any $y \in C$, $f^{n'_i}(y) \to g_1(y) = x_1$ and $f^{n''_i}(y) \to g_2(y) = x_2$, i.e., $f^{n'_i}(x, y) \to (x_1, x_1)$ and $f^{n''_i}(x, y) \to (x_1, x_2)$. Particularly, we have $\liminf d(f^n(x), f^n(y)) = 0$, $\limsup d(f^n(x), f^n(y)) \geqslant \delta$. This ends the proof. □

Note that it is not difficult to show that any minimal weakly mixing system is ST chaotic, it remains open if weak mixing implies ST chaoticity.

3. Weak disjointness

For a dynamical property P, denote by $(X, f) \in P$ the statement that (X, f) has property P. Thus P also stands for the set of all systems having P. If two TDS (X, f) and (Y, g) are weakly disjoint, we write $(X, f) \curlywedge (Y, g)$. And if P is a property, we write $P^\curlywedge$ for the set of all systems weakly disjoint from every $(X, f) \in P$. For a dynamical property P stronger than transitivity it is easy to check that the following basic facts [8]:

$$P_1 \subset P_2 \implies P_2^\curlywedge \subset P_1^\curlywedge, \tag{3.1}$$

$$\bigl(P^\curlywedge\bigr)^\curlywedge \supset P, \tag{3.2}$$

$$P^\curlywedge = \bigl(\bigl(P^\curlywedge\bigr)^\curlywedge\bigr)^\curlywedge. \tag{3.3}$$

Two dynamical properties P_1 and P_2 are *symmetrically dual* if $P_1^\curlywedge = P_2$ and $P_2^\curlywedge = P_1$. By (3.3) for any property stronger than transitivity, $P^\curlywedge$ and $P^{\curlywedge\curlywedge}$ are symmetrically dual. Generally speaking, $P^\curlywedge$ cannot be described explicitly. Thus we are interested in the following question: for which dynamical property P, we have $P^{\curlywedge\curlywedge} = P$ and both P and $P^\curlywedge$ can be described explicitly.

As the question is too general we will restrict our attention to the case when $P = \mathcal{F}$-transitivity ($\mathcal{F}$-trans, for short). The following theorem is very much related to this question. Note that the theorem is first proved by Weiss [12] in some special class and then is generalized to the following form by Akin and Glasner [2].

Weiss–Akin–Glasner Theorem. *Let $\mathcal{F}$ be a proper, translation invariant, thick family. A dynamical system is $k\mathcal{F}$-transitive if and only if it is weakly disjoint from every $\mathcal{F}$-transitive system.*

Proof. We only need to show if (X, f) is not $k\mathcal{F}$-transitive then there exists a $\mathcal{F}$-transitive system (Y, g) such that (X, f) and (Y, g) are not weakly disjoint. As (X, f) is not $k\mathcal{F}$-transitive, there is an opene set $U \subset X$ with $N_f(U, U) \notin k\mathcal{F}$. So $A = \mathbb{Z}_+ \setminus N_f(U, U) \in \mathcal{F}$, and by W-AG Lemma there is $\mathcal{F}$-transitive system (Y, g) and an opene set $V \subset Y$ such that $N_g(V, V) = A \cup \{0\}$. Hence $N_{f\times g}(U \times V, U \times V) = N_f(U, U) \cap N_g(V, V) = \{0\} \notin \mathcal{B}$. That is, (X, f) and (Y, g) are not weakly disjoint. □

Let WM =weak mixing and $TE = k\tau\mathcal{B}$-trans. Using W-AG Theorem we have the following simple observation.

Proposition 3.1. *Let P be a dynamical property. Then*

(1) *There is no P such that $P = P^{\curlywedge}$.*
(2) *If $P^{\curlywedge} \subset P$, then $P^{\curlywedge} \subset WM$.*

Proof. First we show (2). Let $(X, f) \in P^{\curlywedge}$. Then (X, f) is weakly disjoint from any system from P. This implies that $(X, f) \curlywedge (X, f)$. Thus (2) holds.

Now assume that P is a dynamical property such that $P = P^{\curlywedge}$. Then we have $P \subset WM$. Thus by (3.1) and W-AG Theorem

$$WM \supset P = P^{\curlywedge} \supset WM^{\curlywedge} = TE,$$

But this inclusion is false since any nontrivial, equicontinuous, minimal system is TE but not WM. This ends the proof of (1). □

Recall a system (X, f) is $\mathcal{F}$-transitive iff it is $r\mathcal{F}$-transitive. For a proper translation invariant thick family $\mathcal{F}$, we have $\mathcal{F} = r\mathcal{F}$ and by W-AG Theorem

$$(\mathcal{F}\text{-trans})^{\curlywedge} = k\mathcal{F}\text{-trans} \tag{3.4}$$

and

$$(\mathcal{F}\text{-trans})^{\curlywedge\curlywedge} = (k\mathcal{F}\text{-trans})^{\curlywedge} \supset \mathcal{F}\text{-trans}. \tag{3.5}$$

To prove Theorem 3.3 we need

Lemma 3.2. *Let $\mathcal{F}$ be a proper family, then $(\mathcal{F}$-trans$)^{\curlywedge} = kr\mathcal{F}$-trans. Consequently, if $r\mathcal{F} = \mathcal{F}$, then $(\mathcal{F}$-trans$)^{\curlywedge} = k\mathcal{F}$-trans.*

Proof. Obviously $kr\mathcal{F}$-trans $\subset (r\mathcal{F}\text{-trans})^{\curlywedge} = (\mathcal{F}\text{-trans})^{\curlywedge}$. Now we show $(r\mathcal{F}\text{-trans})^{\curlywedge} \subset kr\mathcal{F}$-trans. Assume (X, f) is in $(r\mathcal{F}\text{-trans})^{\curlywedge}$, then it is weakly disjoint from every $\mathcal{F}$-trans system. For every opene subsets U, V of X, and for every $\mathcal{F}$-transitive system (Y, g) and every opene subsets U', V' of Y, $N_f(U, V) \cap N_g(U', V') \neq \emptyset$. By the definition of $r\mathcal{F}$

we have $N_f(U,V)\cap A\neq\emptyset$ for every $A\in r\mathcal{F}$. Hence $N_f(U,V)\in kr\mathcal{F}$, i.e., (X,f) is $kr\mathcal{F}$-transitive. Thus

$$(r\mathcal{F}\text{-trans})^{\curlywedge}=(\mathcal{F}\text{-trans})^{\curlywedge}=kr\mathcal{F}\text{-trans}=rkr\mathcal{F}\text{-trans}. \qquad \square$$

Theorem 3.3. *Let $\mathcal{F}$ be a proper family. Then*

(1) $((\mathcal{F}\text{-}trans)^{\curlywedge})^{\curlywedge}=\mathcal{F}\text{-}trans$ *iff* $r\mathcal{F}=rkrkr\mathcal{F}$.
(2) $kr\mathcal{F}$*-trans and* $krkr\mathcal{F}$*-trans are symmetrically dual, i.e.* $(kr\mathcal{F}\text{-}trans)^{\curlywedge}=krkr\mathcal{F}\text{-}trans$ *and* $(krkr\mathcal{F}\text{-}trans)^{\curlywedge}=kr\mathcal{F}\text{-}trans$.
(3) *Let $\mathcal{F}$ be a proper invariant thick family. Then* $((\mathcal{F}\text{-}trans)^{\curlywedge})^{\curlywedge}=\mathcal{F}\text{-}trans$ *if and only if* $\mathcal{F}=rkrk\mathcal{F}$.

Proof. (1) By Lemma 3.2

$$(\mathcal{F}\text{-trans})^{\curlywedge}=kr\mathcal{F}\text{-trans}=rkr\mathcal{F}\text{-trans}.$$

Hence

$$\big((\mathcal{F}\text{-trans})^{\curlywedge}\big)^{\curlywedge}=(kr\mathcal{F}\text{-trans})^{\curlywedge}=rkrkr\mathcal{F}\text{-trans}.$$

Thus $((\mathcal{F}\text{-trans})^{\curlywedge})^{\curlywedge}=\mathcal{F}$-trans iff $r\mathcal{F}=rkrkr\mathcal{F}$.

(2) As $(\mathcal{F}\text{-trans})^{\curlywedge}=kr\mathcal{F}\text{-trans}=rkr\mathcal{F}$-trans, the result follows from (3.3).

(3) By W-AG Lemma $r\mathcal{F}=\mathcal{F}$. Thus (3) follows from (1). $\square$

For a proper invariant thick family $\mathcal{F}$, if $(\mathcal{F}\text{-trans})^{\curlywedge\curlywedge}=\mathcal{F}$-trans, then it is necessary that $(\mathcal{F}\text{-trans})^{\curlywedge\curlywedge}=\mathcal{F}\text{-trans}\subset WM$. If $\mathcal{F}$ is the family of thick sets, then it is known that $(\mathcal{F}\text{-trans})^{\curlywedge\curlywedge}$ is strictly weaker than *WM* [8]. Thus we further restrict our attention to invariant filter. To do this we need

Lemma 3.4. *Let $\mathcal{F}$ be a proper invariant thick family with $\mathcal{F}\subset k\mathcal{F}$. Then*

(1) $(k\mathcal{F}\text{-}trans)^{\curlywedge}=(\mathcal{F}\text{-}trans)^{\curlywedge\curlywedge}\subset WM$.
(2) $rkrk\mathcal{F}=\tau krk\mathcal{F}$.

Proof. (1) As $\mathcal{F}\subset k\mathcal{F}$ and $\mathcal{F}$ is invariant thick, by W-AG Theorem we have

$$(\mathcal{F}\text{-trans})^{\curlywedge\curlywedge}=(k\mathcal{F}\text{-trans})^{\curlywedge}\subset(\mathcal{F}\text{-trans})^{\curlywedge}=k\mathcal{F}\text{-trans}.$$

This means $(\mathcal{F}\text{-trans})^{\curlywedge\curlywedge}=(k\mathcal{F}\text{-trans})^{\curlywedge}\subset WM$ by Proposition 3.1(2).

(2) By (1) and Lemma 3.2 $(k\mathcal{F}\text{-trans})^{\curlywedge}=krk\mathcal{F}\text{-trans}\subset WM$. By Theorem 2.2 we have $(k\mathcal{F}\text{-trans})^{\curlywedge}=\tau krk\mathcal{F}$-trans. As $(k\mathcal{F}\text{-trans})^{\curlywedge}=rkrk\mathcal{F}$-trans, $rkrk\mathcal{F}=\tau krk\mathcal{F}$. $\square$

Recall that for families $\mathcal{F}_1$ and $\mathcal{F}_2$, $\mathcal{F}_1\cdot\mathcal{F}_2=\{F_1\cap F_2 \mid F_1\in\mathcal{F}_1,\ F_2\in\mathcal{F}_2\}$. Thus we have $\mathcal{F}_1\cup\mathcal{F}_2\subset\mathcal{F}_1\cdot\mathcal{F}_2$. It is easy to check that

$$\mathcal{F}_1\cdot\mathcal{F}_2\subset\mathcal{F} \iff \mathcal{F}_1\cdot k\mathcal{F}\subset k\mathcal{F}_2. \tag{3.6}$$

Hence if $\mathcal{F}$ is a filter then $\mathcal{F}=\mathcal{F}\cdot\mathcal{F}\subset\mathcal{F}\cdot k\mathcal{F}=k\mathcal{F}$.

Now for a proper invariant filter we get

Theorem 3.5. *For a proper invariant filter $\mathcal{F}$, $((\mathcal{F}\text{-trans})^{\curlywedge})^{\curlywedge} = \mathcal{F}\text{-trans}$ iff $\tau k r k\mathcal{F} = \mathcal{F}$.*

Proof. For a filter $\mathcal{F}$, $\mathcal{F} \subset k\mathcal{F}$. And since $\mathcal{F}$ is an invariant filter, it is invariant thick. By Theorem 3.3 and Lemma 3.4, the result follows. □

In Section 4 we will give an invariant filter for which $\tau k r k\mathcal{F} \neq \mathcal{F}$.

Generally it is very difficult to compute $r\mathcal{F}$ for a family $\mathcal{F}$. Now we give an easier checking condition. To do this we need a lemma [3, Proposition 2.7] and Proposition 3.7.

Lemma 3.6. *If $\mathcal{F}$ is a filter, then $\tau\mathcal{F}$ and $\tau k\tau k\tau\mathcal{F}$ are filters and $\tau\mathcal{F} \subset \tau k\tau k\tau\mathcal{F}$. Moreover, $\tau k\tau k\tau k\tau\mathcal{F} = \tau k\tau\mathcal{F}$. If in addition $\mathcal{F}$ is invariant and consequently thick, then $\tau k\tau k\mathcal{F} \cdot \tau k\mathcal{F} = \tau k\mathcal{F}$.*

Now we show:

Proposition 3.7. *If $\mathcal{F}$ is an invariant proper family and satisfies $\mathcal{F} \cdot \tau k\mathcal{F} = \mathcal{F}$, then*

(1) *$\tau k\mathcal{F} = k(\mathcal{F} \cdot k\mathcal{F})$ is a filter.*
(2) *If in addition $\mathcal{F}$ is thick, then $\tau k\mathcal{F}$-trans and $\mathcal{F} \cdot k\mathcal{F}$-trans are symmetrically dual.*

Proof. (1) The proof is similar to the proof of Proposition 2.9 of [3]. First we mention a fact: for any proper family $\mathcal{F}$, $k(\mathcal{F} \cdot k\mathcal{F}) \subset \mathcal{F} \cap k\mathcal{F}$ is a filter and is the largest family $\mathcal{F}'$ satisfying $\mathcal{F} \cdot \mathcal{F}' \subset \mathcal{F}$ [3, Proposition 2.1]. Hence by $\mathcal{F} \cdot \tau k\mathcal{F} = \mathcal{F}$, we have $\tau k\mathcal{F} \subset k(\mathcal{F} \cdot k\mathcal{F}) \subset k\mathcal{F}$. As $\mathcal{F}$ is an invariant family, so is $k(\mathcal{F} \cdot k\mathcal{F})$. Thus $k(\mathcal{F} \cdot k\mathcal{F}) = \tau(k(\mathcal{F} \cdot k\mathcal{F})) \subset \tau k\mathcal{F}$. The result follows.

(2) By (1) and W-AG Theorem we have $(\tau k\mathcal{F}\text{-trans})^{\curlywedge} = k\tau k\mathcal{F}\text{-trans} = k(k(\mathcal{F} \cdot k\mathcal{F}))\text{-trans} = \mathcal{F} \cdot k\mathcal{F}\text{-trans}$ immediately.

Now we show $(\mathcal{F} \cdot k\mathcal{F}\text{-trans})^{\curlywedge} \subset \tau k\mathcal{F}\text{-trans}$. As $\mathcal{F}\text{-trans} \subset \mathcal{F} \cdot k\mathcal{F}\text{-trans}$ and $k\mathcal{F}\text{-trans} \subset \mathcal{F} \cdot k\mathcal{F}\text{-trans}$, we have

$$(\mathcal{F} \cdot k\mathcal{F}\text{-trans})^{\curlywedge} \subset (\mathcal{F}\text{-trans})^{\curlywedge} \quad \text{and} \quad (\mathcal{F} \cdot k\mathcal{F}\text{-trans})^{\curlywedge} \subset (k\mathcal{F}\text{-trans})^{\curlywedge}.$$

By W-AG Theorem $(\mathcal{F}\text{-trans})^{\curlywedge} = k\mathcal{F}\text{-trans}$. So $(\mathcal{F} \cdot k\mathcal{F}\text{-trans})^{\curlywedge} \subset k\mathcal{F}\text{-trans} \cap (k\mathcal{F}\text{-trans})^{\curlywedge}$. Hence $(\mathcal{F} \cdot k\mathcal{F}\text{-trans})^{\curlywedge} \subset k\mathcal{F}\text{-trans} \cap WM = \tau k\mathcal{F}\text{-trans}$ (by Theorem 2.2).

As $\tau k\mathcal{F} = k(\mathcal{F} \cdot k\mathcal{F})$, $\tau k\mathcal{F}\text{-trans} \subset (\mathcal{F} \cdot k\mathcal{F}\text{-trans})^{\curlywedge}$. Thus $(\mathcal{F} \cdot k\mathcal{F}\text{-trans})^{\curlywedge} = \tau k\mathcal{F}\text{-trans}$. □

By Lemma 3.6 and Proposition 3.7 we have the following theorem immediately:

Theorem 3.8. *Let $\mathcal{F}$ be a proper invariant filter. Then*

(1) *$\tau k\tau k\mathcal{F}$-trans and $\tau k\mathcal{F} \cdot k\tau k\mathcal{F}$-trans are symmetrically dual.*
(2) *$\mathcal{F}\text{-trans} \subset (k\mathcal{F}\text{-trans})^{\curlywedge} \subset \tau k\tau k\mathcal{F}\text{-trans}$. Consequently, if $\mathcal{F} = \tau k\tau k\mathcal{F}$, then $\mathcal{F}$-trans and $k\mathcal{F}$-trans are symmetrically dual, i.e., a system is $\mathcal{F}$-trans iff it is weakly disjoint from any $k\mathcal{F}$-trans system, and a system is $k\mathcal{F}$-trans iff it is weakly disjoint from any $\mathcal{F}$-trans system.*

(3) *If $\mathcal{F}$ is an invariant filter with $\tau k\tau k\mathcal{F} = \mathcal{F}$, then $\tau k\tau\mathcal{B} \subset \mathcal{F}$. That is, $\tau k\tau\mathcal{B}$ is the smallest filter with the property.*

Proof. (1) Let $\mathcal{F}_1 = \tau k\mathcal{F}$. Then

$$\mathcal{F}_1 \cdot \tau k\mathcal{F}_1 = \tau k\mathcal{F} \cdot \tau k\tau k\mathcal{F} = \tau k\mathcal{F} = \mathcal{F}_1.$$

Applying Proposition 3.7(1) we get the result.

(2) As $\mathcal{F} \subset \tau k\tau k\mathcal{F}$ (Lemma 3.6), we have $k\mathcal{F}$-trans $= (\mathcal{F}\text{-trans})^{\curlywedge} \supset (\tau k\tau k\mathcal{F}\text{-trans})^{\curlywedge}$. Thus $\mathcal{F}$-trans $\subset (k\mathcal{F}\text{-trans})^{\curlywedge} \subset \tau k\tau k\mathcal{F}$-trans. If $\mathcal{F} = \tau k\tau k\mathcal{F}$, then $(k\mathcal{F}\text{-trans})^{\curlywedge} = \mathcal{F}$-trans and $(\mathcal{F}\text{-trans})^{\curlywedge} = k\mathcal{F}$-trans.

(3) As $k\mathcal{F} \subset \mathcal{B}$, we have $\tau k\tau\mathcal{B} \subset \tau k\tau k\mathcal{F} = \mathcal{F}$. □

Applying the above corollary to the case when $\mathcal{F} = k\mathcal{B}$ or $\mathcal{F} = \tau k\tau\mathcal{B}$, we get the following corollary which appeared in [8].

Corollary 3.9. *$\tau k\tau\mathcal{B}$-trans and $k\tau k\tau\mathcal{B}$-trans are symmetrically dual.*

Generally for an invariant thick family $\mathcal{F}$, $\tau k\mathcal{F}$ is not necessarily a filter. If it is we have

Proposition 3.10. *If $\mathcal{F}$ is a proper invariant thick family, then $\tau k\tau k\tau k\mathcal{F} = \tau k\mathcal{F}$. If in addition $\tau k\mathcal{F}$ is a filter, then $(k\tau k\mathcal{F}\text{-trans})^{\curlywedge} = \tau k\mathcal{F}$-trans and $(\tau k\mathcal{F}\text{-trans})^{\curlywedge} = k\tau k\mathcal{F}$-trans.*

Proof. As $\tau k\mathcal{F} \subset k\mathcal{F}$, we have $k\tau k\mathcal{F} \supset \mathcal{F}$ and $\tau k\tau k\mathcal{F} \supset \tau\mathcal{F} = \mathcal{F}$. Replacing $\mathcal{F}$ by $\tau k\mathcal{F}$, we get $\tau k\tau k\tau k\mathcal{F} \supset \tau k\mathcal{F}$. And by $\tau k\tau k\mathcal{F} \supset \mathcal{F}$, we have $k\tau k\tau k\mathcal{F} \subset k\mathcal{F}$ and hence $\tau k\tau k\tau k\mathcal{F} \subset \tau k\mathcal{F}$. Thus $\tau k\tau k\tau k\mathcal{F} = \tau k\mathcal{F}$.

If $\tau k\mathcal{F}$ is a filter, then by Theorem 3.8(2) we have $(k\tau k\mathcal{F}\text{-trans})^{\curlywedge} = \tau k\mathcal{F}$-trans and $(\tau k\mathcal{F}\text{-trans})^{\curlywedge} = k\tau k\mathcal{F}$-trans. □

There are some questions concerning Theorem 3.8. The first one is that if $\mathcal{F}$ is an invariant filter, then whether it is true that $(k\mathcal{F}\text{-trans})^{\curlywedge} = \mathcal{F}$-trans implies $\mathcal{F} = \tau k\tau k\mathcal{F}$? The second one is: does Theorem 3.8 really give some other symmetrically dual properties which is not the one in Corollary 3.9?

Now we show the answer to the first question is negative, and in Section 4 we will show that the answer to the second question is positive.

Theorem 3.11. *There exists an invariant filter $\mathcal{G}$ such that $((\mathcal{G}\text{-trans})^{\curlywedge})^{\curlywedge} = \mathcal{G}$-trans but $\mathcal{G} \neq \tau k\tau k\mathcal{G}$.*

Proof. Let $\mathcal{F} = k\mathcal{B}$. Then $\mathcal{G} = rkrkr\mathcal{F} = rkr\mathcal{B}$ is an invariant family with $((\mathcal{G}\text{-trans})^{\curlywedge})^{\curlywedge} = \mathcal{G}$-trans by Theorem 3.3. Now we show $\mathcal{G}$ is a filter. For any $A, B \in \mathcal{G}$, by definition there are $kr\mathcal{B}$-transitive systems $(X, f), (Y, g)$ and opene subsets $U, V \subset X$ and $U', V' \subset Y$ such that $N_f(U, V) \subset A$ and $N_g(U', V') \subset B$. For any transitive system (Z, h), as (X, f) is weakly disjoint from (Z, h) we have $(Z \times X, h \times f)$ is transitive. And because (Y, g) also is $kr\mathcal{B}$-transitive, we have $(Z \times X \times Y, h \times f \times g)$ is transitive. Hence $(X \times Y, f \times g)$

is weakly disjoint from any transitive system, i.e., $(X \times Y, f \times g)$ is $kr\mathcal{B}$-transitive. Thus $N_f(U, V) \cap N_g(U', V') \in rkr\mathcal{B} = \mathcal{G}$. So $A \cap B \in \mathcal{G}$ for any $A, B \in \mathcal{G}$, i.e., $\mathcal{G}$ is a filter.

Now we show $\tau k\mathcal{G} = \tau\mathcal{B}$, which implies $\mathcal{G} \neq \tau k\tau k\mathcal{G}$. As $\tau k\mathcal{G}$-trans $= WM \cap k\mathcal{G}$-trans and $k\mathcal{G}$-trans $= r\mathcal{B}$-trans $= \mathcal{B}$-trans $=$ transitivity, we have $\tau k\mathcal{G}$-trans $= WM$. Hence by W-AG Lemma $\tau k\mathcal{G} = r\tau k\mathcal{G} = \tau\mathcal{B}$. □

By the proof of Theorem 3.11 we have

Corollary 3.12. *A system (X, f) is transitive if and only if it is weakly disjoint from every strongly mixing system and a system which is weakly disjoint from every transitive system need not be strongly mixing.*

In the measure theoretical setting, a system is ergodic iff its product with any *WM* is ergodic, and a system is *WM* iff its product with any *WM* system is *WM*. The facts are not valid in topological setting. In [8] the authors show that for a dynamical system (Y, S), $(Y, S) \times WM \subset WM$ iff (Y, S) is $WM \cap TE$, $(Y, S) \times TE \subset TE$ iff (Y, S) is $WM \cap TE$, $(Y, S) \times WM \cap TE \subset WM \cap TE$ iff (Y, S) is $WM \cap TE$.

Motivated by the facts call a family $\mathcal{F}$ *standard* if (1) a system is $\mathcal{F}$-trans iff its product with any $\mathcal{F}$-trans system is $\mathcal{F}$-trans, (2) a system is $\mathcal{F}$-trans iff its product with any $\tau k\mathcal{F}$-trans system is $\tau k\mathcal{F}$-trans and (3) a system is $\mathcal{F}$-trans iff its product with any $k\tau k\mathcal{F}$-trans system is $k\tau k\mathcal{F}$-trans.

Theorem 3.13. *Let $\mathcal{F}$ be a proper invariant thick family. Then $\mathcal{F}$ is standard iff $\mathcal{F}$ is a filter with $\mathcal{F} = \tau k\tau k\mathcal{F}$.*

Proof. Assume $\mathcal{F}$ is a standard family. As $\mathcal{F}$ be an invariant thick family, using (1) and W-AG Lemma we know that $\mathcal{F}$ is a filter. By Lemma 3.6 we know that $\tau k\tau k\mathcal{F} \cdot \tau k\mathcal{F} = \tau k\mathcal{F}$. Using (2) and W-AG Lemma we get that $\mathcal{F} \supset \tau k\tau k\mathcal{F}$. It follows that $\tau k\tau k\mathcal{F} = \mathcal{F}$ by Lemma 3.6.

Now suppose that $\mathcal{F}$ is a filter with $\mathcal{F} = \tau k\tau k\mathcal{F}$. (1) is obvious. We show (2) now. First if (X_1, T_1) is $\mathcal{F}$-trans and (X_2, T_2) is $\tau k\mathcal{F}$-trans respectively then $(X_1 \times X_2, T_1 \times T_2)$ is $\tau k\mathcal{F}$-trans as $\mathcal{F} \cdot \tau k\mathcal{F} = \tau k\tau k\mathcal{F} \cdot \tau k\mathcal{F} = \tau k\mathcal{F}$. If the product of (X_1, T_1) with any $\tau k\mathcal{F}$-trans system is $\tau k\mathcal{F}$-trans then (X_1, T_1) is $\tau k\mathcal{F}$-trans and $k\tau k\mathcal{F}$-trans by W-AG Theorem. By Theorem 2.2(3) we know that (X_1, T_1) is $\tau k\tau k\mathcal{F}$-trans, i.e., $\mathcal{F}$-trans.

Let (X_1, T_1) and (X_2, T_2) be $\mathcal{F}$-trans and $k\tau k\mathcal{F}$-trans, respectively. To show $(X_1 \times X_2, T_1 \times T_2)$ is $k\tau k\mathcal{F}$-trans we need to show its product with any $\tau k\mathcal{F}$-trans system (X_3, T_3) is transitive. Note that $k\tau k\mathcal{F} \cdot \tau k\mathcal{F} = k\mathcal{F}$ [3, Proposition 2.9]. Thus, $(X_2 \times X_3, T_2 \times T_3)$ is $k\mathcal{F}$-trans. It follows that $(X_1 \times X_2 \times X_3, T_1 \times T_2 \times T_3)$ is transitive.

To finish (3) we need to show that if the product of (X_1, T_1) with any $k\tau k\mathcal{F}$-trans system is $k\tau k\mathcal{F}$-trans, then it is $\mathcal{F}$-trans. First (X_1, T_1) is $k\tau k\mathcal{F}$-trans. Thus $(X_1 \times X_1, T_1 \times T_1)$ is $k\tau k\mathcal{F}$-trans, and consequently weakly mixing. By Theorem 2.2 (X_1, T_1) is $\tau k\tau k\mathcal{F}$-trans $=$ $\mathcal{F}$-trans. □

4. Examples

In this section we will give an example with $\tau krk\mathcal{F}$-trans $= (k\mathcal{F}\text{-trans})^{\curlywedge} \neq \mathcal{F}$, where $\mathcal{F}$ is an invariant filter and an example to show that Theorem 3.8 really gives some other symmetrically dual properties which is not the one in Corollary 3.9. The main idea of the first example is to find a translation invariant thick family $\mathcal{F}'$ with $\mathcal{F}'$-trans $\subset (k\mathcal{F}\text{-trans})^{\curlywedge}$ but $\mathcal{F} \subsetneq \mathcal{F}'$, and then by Theorem 3.8 we have $(k\mathcal{F}\text{-trans})^{\curlywedge} \neq \mathcal{F}$-trans. Now we start to show the example.

Let $\mathcal{D} = \{A \subset \mathbb{Z}_+ \mid d(A) = 1\}$. We will prove $(k\mathcal{D}\text{-trans})^{\curlywedge} \neq \mathcal{D}$-trans. It is easy to see $\mathcal{D}$ is an invariant filter and $k\mathcal{D} = \{A \subset \mathbb{Z}_+ \mid \bar{d}(A) > 0\}$. Set $N_k(A) = \{i \in \mathbb{Z}_+ \mid ik \in A\}$, where A is a subset of $\mathbb{Z}_+$.

Lemma 4.1. *$A \in \mathcal{D}$ if and only if $N_k(A) \in \mathcal{D}$ for every $k \in \mathbb{N}$.*

Proof. Since $N_1(A) = A$, sufficiency is obvious. Now assume $A \in \mathcal{D}$ and there exists some k_0 such that $N_{k_0}(A) \notin \mathcal{D}$, i.e., there are $\{n_i\}_{i=1}^{\infty}$ such that

$$\lim_{i\to\infty} \frac{|\{0, 1, \ldots, n_i - 1\} \cap N_{k_0}(A)|}{n_i} = a < 1.$$

Then

$$\begin{aligned}
&\limsup_{i\to\infty} \frac{|\{0, 1, \ldots, k_0 n_i - 1\} \cap A|}{k_0 n_i} \\
&\quad \leqslant \limsup_{i\to\infty} \frac{n_i(k_0 - 1) + |\{0, 1, \ldots, n_i - 1\} \cap N_{k_0}(A)|}{k_0 n_i} \\
&\quad = \frac{k_0 - 1}{k_0} + \lim_{i\to\infty} \frac{|\{0, 1, \ldots, n_i - 1\} \cap N_{k_0}(A)|}{k_0 n_i} \\
&\quad = \frac{k_0 - 1}{k_0} + \frac{a}{k_0} < 1,
\end{aligned}$$

a contradiction as $A \in \mathcal{D}$. □

Lemma 4.2. *Set $\mathcal{F} = \{A \subset \mathbb{Z}_+ \mid N_{2^k}(A) \in k\mathcal{D}$ for every $k \in \mathbb{Z}_+\}$. Then a transitive system (X, f) is $k\mathcal{D}$-transitive if and only if (X, f) is $\mathcal{F}$-transitive.*

Proof. As $\mathcal{F} \subset k\mathcal{D}$, it remains to show if (X, f) is $k\mathcal{D}$-transitive then it is $\mathcal{F}$-trans. Since a transitive system is $\mathcal{F}$-transitive if and only if it is $\mathcal{F}$-central, i.e., $N(U, U) \in \mathcal{F}$ for each opene set, the result follows from the following claim.

Claim. *If (X, f) is $k\mathcal{D}$-central, then for every opene set U and every $k \in \mathbb{N}$,*

$$N_{2^k}(U) = N_{2^k}\big(N_f(U, U)\big) = \big\{i \mid 2^k i \in N_f(U, U)\big\} \in k\mathcal{D}.$$

Proof. We proceed by induction on k. Firstly let $k = 1$, then $N_2(U) = \frac{2\mathbb{Z}_+ \cap N_f(U,U)}{2}$.

Case 1. If $N_f(U, U) \subset 2\mathbb{Z}_+$, then $\bar{d}(N_2(U)) = \bar{d}(\frac{N_f(U,U)}{2}) \geqslant \bar{d}(N_f(U, U))$. So $N_2(U) \in k\mathcal{D}$.

Case 2. If there is an odd number $a \in N_f(U,U)$, then let $V = U \cap f^{-a}(U) \neq \emptyset$. As (X,f) is $k\mathcal{D}$-central, $N_f(V,V) \in k\mathcal{D}$. It is easy to check for every $n \in N_f(V,V)$ we have $n, n+a \in N_f(U,U)$ and since a is odd either n or $n+a$ is even. Hence the even number in $N_f(U,U)$ appears with positive upper density. Consequently $\bar{d}(N_2(U)) > 0$, i.e., $N_2(U) \in k\mathcal{D}$.

Assume the claim has been established for k, i.e., $N_{2^k}(U) \in k\mathcal{D}$ for every opene set U of X. If $N_{2^k}(U) \subset 2\mathbb{Z}_+$, then

$$N_{2^{k+1}}(U) = \frac{N_{2^k}(U)}{2} \in k\mathcal{D}.$$

If there is an odd number $a \in N_{2^k}(U)$, then similar to above we have $N_{2^{k+1}}(U) \in k\mathcal{D}$. $\square$

Theorem 4.3. *Let*

$$\begin{aligned}\mathcal{F}' &= \left\{A \subset \mathbb{Z}_+\colon \text{ for any } r \in \mathbb{Z}, \text{ there is } k_r \in \mathbb{Z}_+ \text{ with } \left\{i \mid 2^{k_r} i - r \in A\right\} \in \mathcal{D}\right\} \\ &= \left\{A \subset \mathbb{Z}_+\colon \text{ for any } r \in \mathbb{Z}, \text{ there is } k_r \in \mathbb{Z}_+ \text{ with } N_{2^{k_r}}(A+r) \in \mathcal{D}\right\},\end{aligned}$$

where $A + r = \{n \in \mathbb{Z}_+\colon n - r \in A\}$. *Then*

(1) $\mathcal{F}'$ *is an invariant filter and* $\mathcal{D} \subsetneqq \mathcal{F}'$.
(2) $\mathcal{F}'$*-trans* $\subset$ $(k\mathcal{D}$*-trans*$)^{\curlywedge}$, *consequently* $\mathcal{D}$*-trans* $\subsetneqq$ $(k\mathcal{D}$*-trans*$)^{\curlywedge}$.

Proof. (1) First we show $\mathcal{F}'$ is a filter. For any two elements A, B of $\mathcal{F}'$, by definition for every $r \in \mathbb{Z}$ there are $k_r, k_r' \in \mathbb{Z}_+$ such that $N_{2^{k_r}}(A+r) \in \mathcal{D}$ and $N_{2^{k_r'}}(B+r) \in \mathcal{D}$. By Lemma 4.1 we have $N_{2^{k_r+k_r'}}(A+r) \in \mathcal{D}$ and $N_{2^{k_r+k_r'}}(B+r) \in \mathcal{D}$. Thus $N_{2^{k_r+k_r'}}((A \cap B)+r) = N_{2^{k_r+k_r'}}((A+r) \cap (B+r)) = N_{2^{k_r+k_r'}}(A+r) \cap N_{2^{k_r+k_r'}}(B+r) \in \mathcal{D}$. So $A \cap B \in \mathcal{F}'$.

Now we show $\mathcal{F}'$ is invariant, i.e., $g^t(\mathcal{F}') = \mathcal{F}'$ for every $t \in \mathbb{Z}$. For any $A \in \mathcal{F}'$, by definition for any $r \in \mathbb{Z}$ there exists k_r such that $\{i \mid 2^{k_r} i - r \in A\} \in \mathcal{D}$ and hence $\{i \mid 2^{k_r} i - r + t \in A + t\} \in \mathcal{D}$. As $r - t$ runs over $\mathbb{Z}$ when r runs over $\mathbb{Z}$, by definition $A + t \in \mathcal{F}'$. Thus $g^t(\mathcal{F}') = \mathcal{F}'$.

Thus $\mathcal{F}'$ is an invariant filter. By Lemma 4.1, $\mathcal{D} \subset \mathcal{F}'$. Now we show $\mathcal{D} \neq \mathcal{F}'$. An element $A \in \mathcal{F}' \setminus \mathcal{D}$ can be constructed in the following way.

Let $A = \bigcup_{r=-\infty}^{\infty}(2^{k_r}\mathbb{Z}_+ - r)$, where k_r will be fixed later. As $d(2^{k_r}\mathbb{Z}_+ - r) = d(2^{k_r}\mathbb{Z}_+) = \frac{1}{2^{k_r}}$, we have $\bar{d}(A) \leqslant \sum_{r=-\infty}^{\infty} d(2^{k_r}\mathbb{Z}_+ - r) = \sum_{r=-\infty}^{\infty} \frac{1}{2^{k_r}}$. Choose $\{k_r\}$ with $\sum_{r=-\infty}^{\infty} \frac{1}{2^{k_r}} < 1$ (for example, let $k_0 = 1$ and $k_r = k_{-r} = 3r$, $r > 1$, then $\sum_{r=-\infty}^{\infty} \frac{1}{2^{k_r}} = \frac{1}{2} + 2 \times \sum_{r=1}^{\infty} \frac{1}{2^{3r}} = \frac{11}{14} < 1$). Thus $A \in \mathcal{F}' \setminus \mathcal{D}$.

(2) Now we show every $\mathcal{F}'$-transitive system (X,f) is weakly disjoint from any $k\mathcal{D}$-transitive system. We only need to check the case when (X,f) is a homeomorphism (for the general case we pass to the natural extension). Let (Y,g) be a $k\mathcal{D}$-transitive system.

Let U_1, U_2 be opene sets of Y and V_1, V_2 be opene sets of X. Let $n_0 \in N_g(U_1, U_2)$ and $U = U_1 \cap g^{-n_0}(U_2)$. And let V be an opene subset of V_1 and $r \in \mathbb{N}$ with $f^{-r}(V) \subset f^{-n_0}(V_2)$. Then we have

$$
\begin{aligned}
N_{g\times f}(U_1\times V_1, U_2\times V_2) &= N_g(U_1,U_2)\cap N_f(V_1,V_2)\\
&\supset n_0 + N_g(U,U)\cap N_f\big(V_1, f^{-n_0}(V_2)\big)\\
&\supset n_0 + N_g(U,U)\cap N_f\big(V, f^{-r}(V)\big)\\
&= n_0 + N_g(U,U)\cap \big(N_f(V,V)-r\big).
\end{aligned}
$$

As $N_f(V,V)\in\mathcal{F}'$, there is $k_{-r}\in\mathbb{N}$ with $N_{2^{k_{-r}}}(N_f(V,V)-r)\in\mathcal{D}$. By Lemma 4.1 $N_{2^{k_{-r}}}(N_g(U,U))\in k\mathcal{D}$. Hence

$$
\begin{aligned}
&N_{2^{k_{-r}}}\big(N_g(U,U)\cap\big(N_f(V,V)-r\big)\big)\\
&\quad = N_{2^{k_{-r}}}\big(N_g(U,U)\big)\cap N_{2^{k_{-r}}}\big(N_f(V,V)-r\big)\neq\emptyset.
\end{aligned}
$$

Particularly $N_g(U,U)\cap(N_f(V,V)-r)\neq\emptyset$. So $N_{g\times f}(U_1\times V_1,U_2\times V_2)\neq\emptyset$, i.e., $(X,f)\curlywedge(Y,g)$. Thus we have proved $\mathcal{F}'$-trans $\subset(k\mathcal{D})^{\curlywedge}$.

By W-AG Lemma there exists (X,f) which is $\mathcal{F}'$-transitive but not $\mathcal{D}$-transitive. So $\mathcal{D}$-trans $\subsetneqq$ $(k\mathcal{D}$-trans$)^{\curlywedge}$. □

Remark 4.4. The same method can be applied to $k\mathcal{B}$ and $BD_*=\{A\mid d_*(A)=1\}$.

Finally we give the second example.

Theorem 4.5. *$\tau k\tau k\mathcal{D}$ is not the family consisting of thickly syndetic sets, i.e.,*

$$\tau k\tau k\mathcal{D}\neq\tau k\tau\mathcal{B}.$$

Proof. Since $k\mathcal{D}\subset\mathcal{B}$, $\tau k\tau\mathcal{B}\subset\tau k\tau k\mathcal{D}$. By Lemma 3.6, $\mathcal{D}\subset\tau k\tau k\mathcal{D}$ and it is impossible that $\tau k\tau k\mathcal{D}=\tau k\tau\mathcal{B}$ as $\mathcal{D}\nsubseteqq\tau k\tau\mathcal{B}$. □

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